

# ON CONVERGENCE SETS OF FORMAL POWER SERIES

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**ABSTRACT.** The (projective) convergence set of a divergent formal power series  $f(x_1, \dots, x_n)$  is defined to be the image in  $\mathbb{P}^{n-1}$  of the set of all  $x \in \mathbb{C}^n$  such that  $f(x_1 t, \dots, x_n t)$ , as a series in  $t$ , converges absolutely near  $t = 0$ . We prove that every countable union of closed complete pluripolar sets in  $\mathbb{P}^{n-1}$  is the convergence set of some divergent series  $f$ . The (affine) convergence sets of formal power series with polynomial coefficients are also studied. The higher-dimensional results of A. Sathaye, P. Lelong, N. Levenberg and R.E. Molzon, and of J. Ribón are thus generalized.

## 1. INTRODUCTION

A formal power series  $f(x_1, \dots, x_n)$  with coefficients in  $\mathbb{C}$  is said to be convergent if it is absolutely convergent in some neighborhood of the origin in  $\mathbb{C}^n$ . A classical result of Hartogs (see [Hartogs 1906]) states that a series  $f$  converges if and only if it converges along all directions  $\xi \in \mathbb{P}^{n-1}$ , *i.e.*,  $f_\xi(t) := f(\xi_1 t, \dots, \xi_n t)$  converges, as a series in  $t$ , for all  $\xi \in \mathbb{P}^{n-1}$ . This can be interpreted as a formal analog of Hartogs' theorem on separate analyticity. Since a divergent power series still may converge in certain directions, it is natural and desirable to consider the set of all such directions. Following Abhyankar-Moh [Abhyankar and Moh 1970], we define the convergence set of a *divergent* power series  $f$  to be the set of all directions  $\xi \in \mathbb{P}^{n-1}$  such that  $f_\xi(t)$  is convergent. For the case  $n = 2$ , P. Lelong [Lelong 1951] proved that the convergence set of a divergent series  $f(x_1, x_2)$  is an  $F_\sigma$  polar set (*i.e.*  $F_\sigma$  set of vanishing logarithmic capacity) in  $\mathbb{P}^1$ , and moreover, every  $F_\sigma$  polar subset of  $\mathbb{P}^1$  is contained in the convergence set of a divergent series  $f(x_1, x_2)$ . The optimal result was later obtained by A. Sathaye (see [Sathaye 1976]) who showed that the class of convergence sets of divergent power series  $f(x_1, x_2)$  is precisely the class of  $F_\sigma$  polar sets in  $\mathbb{P}^1$ . In this paper we prove that a countable union of

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closed complete pluripolar sets in  $\mathbb{P}^{n-1}$  is the convergence set of some divergent series. This generalizes the results of P. Lelong, Levenberg and Molzon, and Sathaye.

We also study convergence sets of power series of the type  $f(s, t) = \sum_j P_j(s)t^j$  where the coefficients  $P_j(s)$  are polynomials with  $\deg(P_j) \leq j$ , as in [Ribón 2004] and [Pérez-Marco 2000].

Theorems 4.5 and 5.15 are our main theorems, the proofs of which were inspired by [Sathaye 1976], and influenced by the methods developed in [Siciak 1982], [Levenberg and Molzon 1988], and [Ribón 2004].

## 2. TRANSFINITE DIAMETER AND CAPACITY

Let  $\mathbb{Z}_+$  denote the set of nonnegative integers. Let  $n$  be a positive integer. For  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$ , let  $|\alpha| = \alpha_1 + \dots + \alpha_n$ . Let  $\{\alpha(1), \alpha(2), \dots\}$  be the listing of the elements of  $\mathbb{Z}_+^n$  indexed using the lexicographic ordering but with  $|\alpha(i)|$  nondecreasing. Set  $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$  for  $x \in \mathbb{C}^n$  and  $\alpha \in \mathbb{Z}_+^n$ .

Let  $m_k = \binom{n+k}{k}$ , the number of monomials of order up to  $k$ . Let  $l_k = \sum_{q=1}^k q(m_q - m_{q-1}) = n \binom{n+k}{k-1}$ .

For a finite set  $\{s_1, \dots, s_j\}$  of points in  $\mathbb{C}^n$ , let  $V(s_1, \dots, s_j) = \det(s_q^{\alpha(p)})_{1 \leq p, q \leq j}$  be the  $j$ -th Vandermonde determinant. For a compact set  $E$  in  $\mathbb{C}^n$ , let

$$V_j(E) = \sup\{V(s_1, \dots, s_j) : s_1, \dots, s_j \in E\}, \quad d_k(E) = (V_{m_k}(E))^{1/l_k}.$$

The limit  $d(E) := \lim_k d_k(E)$  exists ([Zaharjuta 1975]), and is known as the transfinite diameter of  $E$ .

Let  $\mathcal{P}_k(\mathbb{C}^n)$  be the set of polynomials on  $\mathbb{C}^n$  of degrees  $\leq k$ . For a compact set  $E \subset \mathbb{C}^n$  and  $p \in \mathcal{P}_k(\mathbb{C}^n)$ , set

$$|p|_E = \sup\{|p(z)| : z \in E\},$$

$$L_{k,R}(E) = R^{-1}(\sup\{|p|_{\Delta_R} : p \in \mathcal{P}_k(\mathbb{C}^n), |p|_E \leq 1\})^{1/k},$$

and

$$L_R(E) = \sup_k L_{k,R}(E), \quad c(E) = 1/\overline{\lim}_{R \rightarrow \infty} L_R(E).$$

The quantity  $c(E)$  is called the capacity of  $E$ .

**Lemma 2.1.** *For a compact set  $E$  in  $\mathbb{C}^n$  we have*

- (i)  $c(E) = 0$  if and only if  $d(E) = 0$ , and
- (ii) if  $n = 1$ , then  $c(E) = d(E)$ .

For a proof of (i), see [Levenberg and Taylor 1984]. For a proof of (ii), see [Ahlfors 1973, p. 24].

We need to use the following lemma. It appeared, in different forms, in [Sibony and Wong 1978] and [Sibony 1985]. See, also, [Siciak 1982] and [Zaharjuta 1975].

**Lemma 2.2.** (*Bernstein's inequality*) *Let  $E$  be a compact set in  $\mathbb{C}^n$  with  $c(E) > 0$ . Then there is a positive constant  $C_E$  such that for every polynomial  $p(z) = \sum_{|\alpha| \leq d} a_\alpha z^\alpha$ , we have  $|a_\alpha| \leq C_E^d |p|_E$ .*

### 3. SOME CLASSES OF PLURIPOLAR SETS

Let  $E$  be a Borel subset of  $\mathbb{C}^n$ . (Though we do not mention the word “Borel” each time, all subsets of  $\mathbb{C}^n$  considered in this paper are assumed to be Borel.) The set  $E$  is said to be pluripolar (polar when  $n = 1$ ) if for each point  $x \in E$  there is a nonconstant plurisubharmonic function  $u$  defined in a neighborhood  $U$  of  $x$  in  $\mathbb{C}^n$  such that  $u = -\infty$  on  $E \cap U$ . The set  $E$  is said to be globally pluripolar if there is a nonconstant plurisubharmonic function  $u$  defined on  $\mathbb{C}^n$  such that  $E \subset \{y : u(y) = -\infty\}$ . Josefson's theorem (answering a question of P. Lelong) states that  $E$  is pluripolar if and only if  $E$  is globally pluripolar.

The set  $E$  is said to be complete pluripolar if there is a non-constant plurisubharmonic function  $u$  defined on  $\mathbb{C}^n$  such that  $E = \{y : u(y) = -\infty\}$ . So the set  $\{(0, x_2) \in \mathbb{C}^2 : |x_2| < 1\}$  and its closure are pluripolar, but not complete pluripolar. A countable union of pluripolar sets is pluripolar. So the set of rationals in the interval  $[0, 1]$  is polar. It is not complete polar, because each complete pluripolar set is  $G_\delta$ . In  $\mathbb{C}$  each  $G_\delta$  polar set is complete polar, which is Deny's theorem (see [Deny 1947]).

Following Siciak [Siciak 1982, P. 2], we consider families  $L, G, H$  of plurisubharmonic functions:

$$\begin{aligned} L(\mathbb{C}^n) &= \{u \in \text{PSH}(\mathbb{C}^n) : \sup_{x \in \mathbb{C}^n} (u(x) - \ln(1 + |x|)) < \infty\}, \\ G(\mathbb{C}^n) &= \exp(L(\mathbb{C}^n)) = \{e^u : u \in L(\mathbb{C}^n)\}, \\ H(\mathbb{C}^n) &= \{u \in \text{PSH}(\mathbb{C}^n) : u \not\equiv 0, u(\lambda x) = |\lambda|u(x), \forall \lambda \in \mathbb{C}, x \in \mathbb{C}^n\}. \end{aligned}$$

A pluripolar set  $E$  in  $\mathbb{C}^n$  is said to be  $L$ -complete if there is a non-constant  $u \in L(\mathbb{C}^n)$  such that  $E = \{u = -\infty\}$ . A pluripolar set  $F$  in  $\mathbb{C}^n$  is said to be  $H$ -complete if there is a  $w \in H(\mathbb{C}^n)$  such that  $F = \{x : w(x) = 0\}$ .

It follows from the one-to-one correspondence (see [Siciak 1982, Prop. 2.7])

$$(1) \quad H(\mathbb{C} \times \mathbb{C}^n) \ni f(x_0, x) \mapsto \log f(1, x) \in L(\mathbb{C}^n)$$

between functions of the class  $H$  of  $n + 1$  variables and the functions of the class  $L$  of  $n$  variables that each  $H$ -complete pluripolar set in

$\mathbb{C} \times \mathbb{C}^n$  induces a unique  $L$ -complete pluripolar set in  $\mathbb{C}^n$ , and that each  $L$ -complete pluripolar set in  $\mathbb{C}^n$  is induced by a (not necessarily unique)  $H$ -complete pluripolar set in  $\mathbb{C} \times \mathbb{C}^n$ .

Let  $|x| = (|x_1|^2 + \dots + |x_n|^2)^{1/2}$ . Recall that  $\mathcal{P}_k(\mathbb{C}^n)$  is the set of polynomials on  $\mathbb{C}^n$  of degrees  $\leq k$ . Let  $\mathcal{H}_k(\mathbb{C}^n)$  be the set of homogeneous polynomials on  $\mathbb{C}^n$  of degree  $k$ . Let

$$Q(\mathbb{C}^n) = \{(p, k) : k \in \mathbb{N}, p \in \mathcal{P}_k(\mathbb{C}^n)\},$$

$$|(p(x), k)| = |p(x)|^{1/k}, \quad \|(p(x), k)\| = |p(x)|^{1/k} / (1 + |x|^2)^{1/2},$$

$$|(p, k)|_K = \sup\{|(p(x), k)| : x \in K\},$$

and

$$\|(p, k)\|_K = \sup\{\|(p(x), k)\| : x \in K\}, \quad \|(p, k)\| = \|(p, k)\|_{\mathbb{C}^n}.$$

Let

$$\Gamma(\mathbb{C}^n) = \{(h, k) : k \in \mathbb{N}, h \in \mathcal{H}_k(\mathbb{C}^n)\},$$

$$\|(h(x), k)\| = |h(x)|^{1/k} / |x|,$$

and

$$\|(h, k)\|_K = \sup\{\|(h(x), k)\| : x \in K, x \neq 0\}, \quad \|(h, k)\| = \|(h, k)\|_{\mathbb{C}^n}.$$

**Definition 3.1.** Let  $F \subset \mathbb{C}^n$ ,  $F \neq \emptyset$ ,  $x \in \mathbb{C}^n$ , and  $0 \leq r \leq 1$ . Define

$$\tau_H(x, F, r) = \inf\{\|(h, k)\|_F : (h, k) \in \Gamma(\mathbb{C}^n), |h(x)|^{1/k} \geq r|x|, \|(h, k)\| \leq 1\},$$

$$T_H(x, F) = \sup\{r : \tau_H(x, F, r) = 0\},$$

$$\tau_L(x, F, r) = \inf\{\|(h, k)\|_F : (h, k) \in Q(\mathbb{C}^n), \|(h(x), k)\| \geq r, \|(h, k)\| \leq 1\},$$

$$T_L(x, F) = \sup\{r : \tau_L(x, F, r) = 0\}.$$

For the empty set, we define  $\tau_H(x, \emptyset, r) = \tau_L(x, \emptyset, r) = 0$ , and  $T_H(x, \emptyset) = T_L(x, \emptyset) = 1$ .

It is clear that if  $E \subset F$ , then  $\tau_L(x, E, r) \leq \tau_L(x, F, r)$  and  $T_L(x, E) \geq T_L(x, F)$ .

**Lemma 3.2.** Let  $u \in H(\mathbb{C}^n)$  be continuous, with  $\sup\{u(x) : |x| = 1\} = 1$ , and let  $F = \{x \in \mathbb{C}^n : u(x) = 0\}$ . Then for each  $x \in \mathbb{C}^n \setminus F$ ,  $T_H(x, F) \geq u(x)/|x|$ .

*Proof.* Fix  $x \in \mathbb{C}^n \setminus F$ . Then  $x \neq 0$ , since  $u(0) = 0$ . Let  $r$  be a positive number such that  $r < u(x)/|x| \leq 1$ , and let  $\delta \in (0, r)$ . Let  $\phi(x) = \max(u(x), \delta|x|)$ . Then  $\phi$  is a continuous function in  $H(\mathbb{C}^n)$ . By [Siciak 1982, Prop. 2.10], for all  $y \in \mathbb{C}^n$ ,

$$\phi(y) = \sup\{|h(y)|^{1/k} : (h, k) \in \Gamma(\mathbb{C}^n), |h(z)|^{1/k} \leq \phi(z) \forall z \in \mathbb{C}^n\}.$$

Thus there is an  $(h, k) \in \Gamma(\mathbb{C}^n)$  such that  $|h(z)|^{1/k} \leq \phi(z) \forall z \in \mathbb{C}^n$ , and  $r|x| < |h(x)|^{1/k} \leq \phi(x)$ , and therefore

$$|h(x)|^{1/k} > r|x|, \|(h, k)\| \leq 1, \|(h, k)\|_F \leq \delta.$$

It follows that  $\tau_H(x, F, r) \leq \delta$ . Hence  $\tau_H(x, F, r) = 0$  for each  $r < u(x)/|x|$ . Therefore,  $T_H(x, F) \geq u(x)/|x|$ .  $\square$

Since (1) is a one-to-one correspondence, each  $L$ -complete pluripolar set in  $\mathbb{C}^n$  is related to a  $H$ -complete pluripolar set in  $\mathbb{C} \times \mathbb{C}^n$ .

**Proposition 3.3.** *Let  $E = \{v = 0\}$  be an  $L$ -complete pluripolar set in  $\mathbb{C}^n$  with  $v \in L(\mathbb{C}^n)$  such that the function  $u(x_0, x) := |x_0| \exp(v(x/x_0))$  defined on  $\{x_0 \neq 0\}$  extends to be a continuous function on  $\mathbb{C} \times \mathbb{C}^n$ . Then for each  $x \in \mathbb{C}^n \setminus E$ ,  $T_L(x, E) \geq (1 + |x|^2)^{-1/2} \exp(v(x))$ .*

*Proof.* This is a consequence of the previous lemma.  $\square$

**Lemma 3.4.** *Let  $E = \{g = 0\}$  be a closed  $L$ -complete pluripolar set in  $\mathbb{C}^n$  with  $g \in G(\mathbb{C}^n)$  such that  $\sup\{(1 + |y|^2)^{-1/2} g(y) : y \in \mathbb{C}^n\} = 1$ . Then for each  $x \in \mathbb{C}^n \setminus E$ , and each compact set  $K$ ,  $T_L(x, E \cap K) \geq (1 + |x|^2)^{-1/2} g(x)$ .*

*Proof.* If  $E \cap K = \emptyset$ , then the desired inequality clearly holds since  $T_L(x, \emptyset) = 1$ . Fix  $x \in \mathbb{C}^n \setminus E$  and a compact set  $K$  with  $E \cap K \neq \emptyset$ . Let  $r > 0$  be such that  $r < (1 + |x|^2)^{-1/2} g(x)$ . Let  $\eta$  be a positive number with  $\eta < r$ . Let  $\lambda$  be a positive number that is less than the distance between the closed set  $\{y : g(y) \geq \eta\}$  and the compact set  $K \cap E$ , and that is so small that

$$(2) \quad (\lambda + \eta)^{1-\lambda} < \sqrt{\eta}.$$

Let

$$\omega(y) = \begin{cases} c_n \exp(-1/(1 - |y|^2)), & \text{if } |y| < 1, \\ 0, & \text{if } |y| \geq 1, \end{cases} \quad \int \omega(y) dy = 1.$$

For  $\mu > 0$ , let  $g_\mu(y) = \int g(y + \mu z) \omega(z) dz$ . Then  $g_\mu \in G(\mathbb{C}^n)$ ,  $g_\mu$  is  $C^\infty$  and positive, and  $g_\mu \downarrow g$  as  $\mu \downarrow 0$ . If  $y \in K \cap E$ , and if  $|z| < 1$ , then  $y + \lambda z \notin \{y : g(y) \geq \eta\}$ , and hence  $g(y + \lambda z) < \eta$ . It follows that  $g_\lambda(y) = \int g(y + \lambda z) \omega(z) dz < \int \eta \omega(z) dz = \eta$ . For each  $y \in \mathbb{C}^n$ ,

$$g_\lambda(y) \leq \int (1 + |y + \lambda z|^2)^{1/2} \omega(z) dz \leq (1 + (|y| + \lambda)^2)^{1/2},$$

and hence

$$g_\lambda(y) \leq (1 + \lambda)(1 + |y|^2)^{1/2}.$$

As in [Siciak 1982, p. 17], we define a function  $\phi_\lambda \in H(\mathbb{C} \times \mathbb{C}^n)$  by

$$\phi_\lambda(y_0, y) = \begin{cases} |y_0|(\lambda + g_\lambda(y/y_0))^{1-\lambda} + \lambda(|y_0|^2 + |y|^2)^{1/2}, & \text{if } y_0 \neq 0, \\ \lambda|y|, & \text{if } y_0 = 0, \end{cases}$$

and define a function  $\psi_\lambda \in G(\mathbb{C}^n)$  by

$$\psi_\lambda(y) = \phi_\lambda(1, y) = (\lambda + g_\lambda(y))^{1-\lambda} + \lambda(1 + |y|^2)^{1/2}.$$

Then  $\psi_\lambda$  is  $C^\infty$ , and  $\phi_\lambda$  is continuous.

By [Siciak 1982, Prop. 2.10],

$$\phi_\lambda(y_0, y) = \sup\{|h(y_0, y)|^{1/k}\},$$

where the supremum is taken over all  $(h, k) \in \Gamma(\mathbb{C} \times \mathbb{C}^n)$  such that  $|h(z_0, z)|^{1/k} \leq \phi_\lambda(z_0, z) \forall (z_0, z) \in \mathbb{C} \times \mathbb{C}^n$ . It follows that

$$(3) \quad \psi_\lambda(x) = \sup\{|p(x)|^{1/k} : (p, k) \in Q(\mathbb{C}^n), |p(y)|^{1/k} \leq \psi_\lambda(y) \forall y \in \mathbb{C}^n\}.$$

For all  $y \in \mathbb{C}^n$ ,

$$\begin{aligned} \psi_\lambda(y) &= (\lambda + g_\lambda(y))^{1-\lambda} + \lambda(1 + |y|^2)^{1/2} \\ &\leq (\lambda + (1 + \lambda)(1 + |y|^2)^{1/2})^{1-\lambda} + \lambda(1 + |y|^2)^{1/2} \\ &< (1 + 3\lambda)(1 + |y|^2)^{1/2}. \end{aligned}$$

If  $y \in K \cap E$ , then

$$\begin{aligned} \psi_\lambda(y) &= (\lambda + g_\lambda(y))^{1-\lambda} + \lambda(1 + |y|^2)^{1/2} \\ &\leq (\lambda + \eta)^{1-\lambda} + \lambda(1 + |y|^2)^{1/2} \\ &< \sqrt{\eta} + \lambda(1 + |y|^2)^{1/2} \\ &\leq (\sqrt{\eta} + \lambda)(1 + |y|^2)^{1/2}. \end{aligned}$$

So  $|p(z)|^{1/k} \leq \psi_\lambda(z) \forall z \in \mathbb{C}^n$  implies that

$$(4) \quad \|(p, k)\| \leq 1 + 3\lambda \quad \text{and} \quad \|(p, k)\|_{K \cap E} \leq \sqrt{\eta} + \lambda.$$

For sufficiently small  $\lambda$ ,

$$(\lambda + g_\lambda(x))^{1-\lambda} + \lambda(1 + |x|^2)^{1/2} > (1 + 3\lambda)r(1 + |x|^2)^{1/2},$$

since as  $\lambda$  approaches 0, the difference of the left side minus the right side tends to  $g(x) - r(1 + |x|^2)^{1/2} > 0$ . It follows that for sufficiently small  $\lambda$ ,

$$(5) \quad \psi_\lambda(x) > (1 + 3\lambda)r(1 + |x|^2)^{1/2}.$$

By (3), (4) and (5), we have  $\tau(x, E \cap K, r) \leq (1 + 3\lambda)^{-1}(\sqrt{\eta} + \lambda)$ . Letting  $\lambda \rightarrow 0$ , and then  $\eta \rightarrow 0$ , yields that  $\tau(x, E \cap K, r) = 0$ . Since this holds for every  $r < g(x)(1 + |x|^2)^{-1/2}$ , it follows that  $T_L(x, E \cap K) \geq g(x)(1 + |x|^2)^{-1/2}$ .  $\square$

**Definition 3.5.** A pluripolar set  $E$  in  $\mathbb{C}^n$  is said to be  $J$ -complete if for each  $x \in \mathbb{C}^n \setminus E$ , and each compact set  $K$ ,  $T_L(x, E \cap K) > 0$ .

Note that the empty set is  $J$ -complete. Also, it is clear that a  $J$ -complete pluripolar set has to be closed.

**Proposition 3.6.** Every closed  $L$ -complete pluripolar set in  $\mathbb{C}^n$  is  $J$ -complete.

*Proof.* This is a consequence of Lemma 3.4.  $\square$

**Proposition 3.7.** An intersection of  $J$ -complete pluripolar sets in  $\mathbb{C}^n$  is  $J$ -complete. A finite union of  $J$ -complete pluripolar sets in  $\mathbb{C}^n$  is  $J$ -complete.

*Proof.* Let  $\{E_\alpha\}_{\alpha \in \Lambda}$  be a family of  $J$ -complete pluripolar sets in  $\mathbb{C}^n$  and let  $E = \bigcap_{\alpha \in \Lambda} E_\alpha$ . Let  $K$  be a compact set in  $\mathbb{C}^n$  and let  $x \in \mathbb{C}^n \setminus E$ . Then there is a  $\beta \in \Lambda$  such that  $x \notin E_\beta$ . Thus  $T_L(x, E \cap K) \geq T_L(x, E_\beta \cap K) > 0$ . Therefore,  $E$  is  $J$ -complete.

Let  $F_1, \dots, F_m$  be  $J$ -complete pluripolar sets in  $\mathbb{C}^n$  and let  $F = \bigcup_{j=1}^m F_j$ . Let  $K$  be a compact set in  $\mathbb{C}^n$  and let  $x \in \mathbb{C}^n \setminus F$ . Choose a number  $r$  such that  $0 < r < \min_j T_L(x, F_j \cap K)$ . Then  $\tau_L(x, F_j \cap K, r) = 0$  for  $j = 1, \dots, m$ . Let  $\varepsilon > 0$ . Then there are  $(h_j, k_j) \in Q(\mathbb{C}^n)$ ,  $j = 1, \dots, m$ , such that

$$\|(h_j, k_j)\|_{F_j \cap K} < \varepsilon, \quad \|(h_j(x), k_j)\| \geq r, \quad \|(h_j, k_j)\| \leq 1.$$

Raising each  $h_j$  to a suitable power, we may assume that  $k_1 = \dots = k_m = k$ . Let  $h = \prod h_j$ . Then  $(h, mk) \in Q(\mathbb{C}^n)$ , and

$$\|(h, mk)\|_{F \cap K} < \varepsilon^{1/m}, \quad \|(h(x), mk)\| \geq r, \quad \|(h, mk)\| \leq 1.$$

Thus  $\tau_L(x, F \cap K, r) \leq \varepsilon^{1/m}$  for each  $\varepsilon > 0$ . It follows that  $\tau_L(x, F \cap K, r) = 0$  and  $T_L(x, F \cap K) \geq r > 0$ . Therefore,  $F$  is  $J$ -complete.  $\square$

The following theorem is due to A. Saddulaev, Since his book that includes the theorem has not been published, we include his proof here. We are grateful to him for sending us the statement and proof of the theorem, and to B. Fridman for translating an explanation message of A. Saddulaev from Russian to English.

**Theorem 3.8.** Every complete pluripolar set in  $\mathbb{C}^n$  is  $L$ -complete.

*Proof.* Suppose that  $E$  is a complete pluripolar set in  $\mathbb{C}^n$ . Let  $u$  be a plurisubharmonic function such that  $E = \{x : u(x) = -\infty\}$ . Choose an increasing sequence  $\{M_j\}$  of positive numbers such that  $\lim M_j = \infty$  and  $M_j \geq \sup_{|z| \leq \exp 2^j} u(z)$ . For each  $j$ , define a function  $v_j$  by

$$v_j(x) = \begin{cases} \max(2^{-j}(M_j^{-1}u(x) - 1), 2^{-j} \log |x| - 1), & \text{if } |x| < \exp 2^j; \\ 2^{-j} \log |x| - 1, & \text{if } |x| \geq \exp 2^j. \end{cases}$$

Since for each  $\zeta$  on the boundary of the ball  $B(0, \exp 2^j)$ ,

$$\limsup_{|x| < \exp 2^j, x \rightarrow \zeta} (2^{-j}(M_j^{-1}u(x) - 1)) \leq 0 = 2^{-j} \log |\zeta| - 1,$$

the function  $v_j$  is plurisubharmonic on  $\mathbb{C}^n$  by the gluing theorem. On each open set with compact closure, all but a finite number of  $v_j$  are non-positive. It follows that the sum  $v(x) := \sum_{j=1}^{\infty} v_j(x)$  is plurisubharmonic (or identically  $-\infty$ ), since the sequence of the partial sums of the series is eventually non-increasing. It is clear that  $v_j(x) \leq 2^{-j} \log^+ x$  for each  $j$ , so that  $v(x) \leq \log^+ x$ . Thus  $v \in L(\mathbb{C}^n)$  (or  $v$  is identically  $-\infty$ ).

Suppose that  $y \in E$ . Then  $u(y) = -\infty$ , and  $v_j(y) = 2^{-j} \log |y| - 1$  for each  $j$ . Thus  $v(y) = -\infty$ .

Now suppose that  $y \in \mathbb{C}^n \setminus E$  so that  $u(y) > -\infty$ . Then  $v_j(y) > -\infty$  for each  $j$ . Since

$$\lim_{j \rightarrow \infty} 2^{-j}(M_j^{-1}u(y) - 1) = 0 > -1 = \lim_{j \rightarrow \infty} (2^{-j} \log |y| - 1),$$

it follows that there is a positive integer  $m = m(y)$  such that

$$2^{-j}(M_j^{-1}u(y) - 1) > -1/2 > (2^{-j} \log |y| - 1), \quad \text{for } j > m.$$

Thus  $y \in B(0, \exp 2^j)$  and  $v_j(y) = 2^{-j}(M_j^{-1}u(y) - 1)$  for  $j > m$ , and therefore

$$\begin{aligned} v(y) &= \sum_{j=1}^m v_j(y) + \sum_{j=m+1}^{\infty} 2^{-j}(M_j^{-1}u(y) - 1) \\ &\geq \sum_{j=1}^m v_j(y) + \sum_{j=1}^{\infty} 2^{-j}(-M_1^{-1}|u(y)| - 1) \\ &= \sum_{j=1}^m v_j(y) + (-M_1^{-1}|u(y)| - 1) > -\infty. \end{aligned}$$

This implies, in particular, that  $v$  is not identically  $-\infty$ . It follows that  $v \in L(\mathbb{C}^n)$  and  $E = \{x : v(x) = -\infty\}$ .  $\square$

**Theorem 3.9.** *Every closed complete pluripolar set in  $\mathbb{C}^n$  is  $J$ -complete.*

*Proof.* This is a consequence of Proposition 3.6 and Theorem 3.8.  $\square$



Let  $E$  be a non-empty compact pluripolar set in  $\mathbb{C}^n$ . Define the extremal function  $\Phi_E : \mathbb{C}^n \rightarrow [0, \infty]$  by  $\Phi_E(x) = \sup\{|p(x)|^{1/k} : (p, k) \in Q(\mathbb{C}^n), |(p, k)|_E \leq 1\}$ . The  $G$ -hull of  $E$  is defined to be  $\hat{E}^G := \{x \in \mathbb{C}^n : \Phi_E(x) < \infty\}$ . The  $G$ -hull of the empty set is defined to be the empty set. Since  $\hat{E}^G = \cup_k \{x : \Phi_E(x) \leq k\}$ , it follows that  $\hat{E}^G$  is an  $F_\sigma$  pluripolar set.

A pluripolar set is said to be  $G$ -complete if it is the  $G$ -hull of a compact pluripolar set. A compact complete pluripolar set  $K$  in  $\mathbb{C}^n$  is  $G$ -complete, since  $\hat{K}^G = K$  (see [Levenberg and Molzon 1988]).

#### 4. CONVERGENCE SETS IN AFFINE SPACES

Consider a series  $f \in \mathbb{C}[s_1, \dots, s_n][[t]]$  of the form  $f(s, t) = \sum_{j=0}^{\infty} P_j(s)t^j$ , where  $P_j(s) = P_j(s_1, \dots, s_n)$  are polynomials of  $n$  variables. Define

$$\text{Conv}(f) = \{s \in \mathbb{C}^n : f(s, t) \text{ converges as a power series in } t\}.$$

Let  $A, B$  be nonnegative integers with  $A > 0$ . A series  $f(s, t) = \sum_j P_j(s)t^j$  is said to be in Class  $(A, B)$  if  $\deg(P_j) \leq Aj + B$ .

It is clear that Class  $(1, 0)$  is a subset of Class  $(A, B)$ . Suppose that  $E = \text{Conv}(f)$  for some  $f$  in Class  $(A, B)$ . Write  $f(s, t) = \sum_j P_j(s)t^j$ . Set  $g(s, t) = t^N f(s, t^N)$ , where  $N = A + B$ . Then  $g$  is in Class  $(1, 0)$  and  $\text{Conv}(g) = \text{Conv}(f)$ . Therefore, the convergence sets for Class  $(A, B)$  are exactly the convergence sets for Class  $(1, 0)$ .

Suppose that  $f(s, t) = \sum_{j=0}^{\infty} P_j(s)t^j$  is in Class  $(1, 0)$  and  $\text{Conv}(f) = \mathbb{C}^n$ . Then, by Hartogs' classical theorem,  $f(s, t)$  converges as a power series in  $n + 1$  indeterminants  $s$  and  $t$ , i.e.,  $f(s, t)$  converges absolutely for  $(s, t)$  in some neighborhood of the origin in  $\mathbb{C}^n \times \mathbb{C}$ . In this case, we say  $f$  is a convergent series. Conversely, if  $\text{Conv}(f) \neq \mathbb{C}^n$ , then  $f(s, t)$  diverges as a power series in  $s$  and  $t$ , i.e.,  $f(s, t)$  converges absolutely in no neighborhood of the origin in  $\mathbb{C}^{n+1}$ . In this case, we say  $f$  is a divergent series.

**Definition 4.1.** A subset  $E$  of  $\mathbb{C}^n$  is said to be a convergence set in  $\mathbb{C}^n$  if  $E = \text{Conv}(f)$  for some divergent series  $f$  of Class  $(1, 0)$ .

**Theorem 4.2.** Let  $E$  be a convergence set in  $\mathbb{C}^n$ . Then  $E$  is a countable union of  $G$ -complete pluripolar sets. Hence  $E$  is an  $F_\sigma$  pluripolar set.

*Proof.* There is divergent series  $f(s, t) = \sum_{j=1}^{\infty} P_j(s)t^j$  of Class  $(1, 0)$  such that  $E = \text{Conv}(f)$ . Put, for  $m = 1, 2, 3, \dots$ ,

$$(6) \quad E_m = \{s \in \mathbb{C}^n : |s| \leq m, |P_j(s)|^{1/j} \leq m, \text{ for } j = 1, 2, \dots\}.$$

Then  $E = \cup E_m$ .

Suppose, if possible, that for some positive integer  $m$ ,  $c(E_m) > 0$ . Then, by Bernstein's inequality (Lemma 2.2), the coefficients  $b_{j\alpha}$  of  $P_j(s) = \sum b_{j\alpha} s^\alpha$  satisfy  $|b_{j\alpha}| \leq (C_{E_m} m)^j$ , where  $C_{E_m}$  is a constant depending only on  $E_m$ . It follows that the series  $f(s, t)$  is convergent, contradicting the hypothesis. Therefore each  $E_m$  is pluripolar, and  $E$  is an  $F_\sigma$  pluripolar set.

Fix a non-empty  $E_m$  and a point  $s \in \hat{E}_m^G$ . Then  $\gamma := \Phi_{E_m}(s) < \infty$ . Then  $|P_j(s)|^{1/j} \leq \gamma m$  for all  $j$ , and hence  $s \in \text{Conv}(f)$ . Thus  $\hat{E}_m^G \subset E$  for all  $m$ . Therefore,  $E = \cup \hat{E}_m^G$ , and  $E$  is a countable union of  $G$ -complete pluripolar sets.  $\square$

**Theorem 4.3.** *Every  $G$ -complete pluripolar set in  $\mathbb{C}^n$  is a convergence set.*

*Proof.* The theorem is proved by following the approach in [Levenberg and Molzon 1988, Theorem 5.6]. Let  $E$  be a non-empty  $G$ -complete pluripolar set in  $\mathbb{C}^n$ . Then  $E = \hat{K}^G$ , where  $K$  is a non-empty compact pluripolar set. Let  $\mathcal{F}_K$  be the collection of members  $(p, k) \in Q(\mathbb{C}^n)$  such that  $k \geq 1$ ,  $p$  has rational coefficients, and  $|(p, k)|_K \leq 1$ . Let  $\{(p_j, k_j)\}$  be an enumeration of  $\mathcal{F}_K$ . Choose a sequence  $\{r_j\}$  of positive integers so that the sequence  $\{r_j k_j\}$  is strictly increasing. Let  $f(s, t) = \sum_{j=1}^{\infty} p_j(s)^{r_j} t^{r_j k_j}$ . Then  $f$  is of Class  $(1, 0)$ .

Suppose  $s \in E$ . Then  $\alpha := \Phi_K(s) < \infty$ . It follows that  $|p_j(s)^{r_j}| \leq \alpha^{r_j k_j}$  for all  $j$ , and hence  $s \in \text{Conv}(f)$ . Therefore,  $E \subset \text{Conv}(f)$ .

We now consider a point  $s \notin E$ . Then  $\Phi_K(s) = \infty$ . For each positive integer  $m$  there is a  $(p, k) \in Q(\mathbb{C}^n)$  such that  $|(p, k)|_K \leq 1$  and  $|(p(s), k)| > m$ , so there is a  $j_m$  such that  $|(p_{j_m}, k_{j_m})|_K \leq 1$  and  $|(p_{j_m}(s), k_{j_m})| > m$ . It follows that the sequence  $\{|(p_j(s)^{r_j}, r_j k_j)|\}$  is unbounded, and  $s \notin \text{Conv}(f)$ . Therefore,  $E = \text{Conv}(f)$ .  $\square$

**Theorem 4.4.** *Let  $E$  be a countable union of  $J$ -complete pluripolar sets in  $\mathbb{C}^n$ . Then  $E$  is a convergence set.*

*Proof.* The set  $E$  can be expressed as  $E = \cup E_m$ , where  $\{E_m\}$  is an ascending sequence of  $J$ -complete pluripolar sets. For each positive integer  $m$ , we shall construct a sequence  $\{(h_{mk}, q_{mk})\}_{k=1}^{\infty}$  in  $Q(\mathbb{C}^n)$  such that

$$(i) \quad |(h_{mk}, q_{mk})|_{\overline{B}_m \cap E_m} \leq 1,$$

$$(ii) \quad \|(h_{mk}, q_{mk})\| \leq m,$$

$$(iii) \quad \cup_{k=1}^{\infty} \{x : |(h_{mk}(x), q_{mk})| > m/2\} \supset \mathbb{C}^n \setminus E_m,$$

where  $\overline{B}_m$  is the closed ball in  $\mathbb{C}^n$  of center 0 and radius  $m$ .

Fix  $m$  and suppose that  $y \in \mathbb{C}^n \setminus E_m$ . Then  $T_L(x, E_m \cap \overline{B}_m) > 0$ . Thus there is a positive number  $r < 1$  such that

$$\inf\{|(p, v)|_{\overline{B}_m \cap E_m} : (p, v) \in Q(\mathbb{C}^n), |(p(y), v)| \geq r, \|(p, v)\| \leq 1\} = 0.$$

Choose a positive rational number  $\beta = a/b < 1$ , where  $a, b$  are positive integers, such that  $(r/m)^\beta > 1/2$ . There is a member  $(p, v)$  of  $Q(\mathbb{C}^n)$  such that

$$|(p, v)|_{E_m \cap \overline{B}_m} < m^{-1/\beta}, \quad |(p(y), v)| \geq r, \quad \|(p, v)\| \leq 1.$$

Let  $h_{(y)}(x) = p(x)^a m^{v(b-a)}$ , and  $q_{(y)} = bv$ . Then  $(h_{(y)}, q_{(y)}) \in Q(\mathbb{C}^n)$ , and  $|(h_{(y)}(x), q_{(y)})| = |(p(x), v)|^\beta m^{1-\beta}$ . We have, for all  $x \in \mathbb{C}^n$ ,

$$|(h_{(y)}(x), q_{(y)})| \leq (1 + |x|^2)^{\beta/2} m^{1-\beta} \leq m(1 + |x|^2)^{1/2},$$

$$|(h_{(y)}, q_{(y)})|_{E_m \cap \overline{B}_m} < m^{-1} m^{1-\beta} = m^{-\beta} \leq 1,$$

and

$$|(h_{(y)}(y), q_{(y)})| \geq r^\beta m^{1-\beta} = (r/m)^\beta m > m/2.$$

Put  $U_y := \{x : |(h_{(y)}(x), q_{(y)})| > m/2\}$ . Then  $U_y$  is an open neighborhood of  $y$ . Since the set  $\mathbb{C}^n \setminus E_m$  is open, the open cover  $\{U_y : y \in \mathbb{C}^n \setminus E_m\}$  of  $\mathbb{C}^n \setminus E_m$  contains a countable subcover  $\{U_{y_k} : k = 1, 2, \dots\}$ . Write  $(h_{mk}, q_{mk}) = (h_{(y_k)}, q_{(y_k)})$ . Then the sequence  $\{(h_{mk}, q_{mk})\}_{k=1}^\infty$  satisfies (i), (ii) and (iii).

Let  $\{(P_\nu, q_\nu)\}$  be a sequence obtained by arranging  $\{(h_{mk}, q_{mk})\}$  as a single sequence. By raising  $\{(P_\nu, q_\nu)\}$  to suitable powers, we assume that  $\{q_\nu\}$  is an increasing sequence. Put  $f(x, t) = \sum_\nu P_\nu(x) t^{q_\nu}$ . Then  $f$  is of Class (1, 0). We shall show that  $E = \text{Conv}(f)$ .

Suppose that  $x \in E$  and  $\nu$  is a positive integer. Then  $x \in \overline{B}_{m_0} \cap E_{m_0}$  for some positive integer  $m_0$ . Now  $(P_\nu, q_\nu) = (h_{mk}, q_{mk})$  for some  $m, k$ . If  $m \geq m_0$ , then  $|(P_\nu(x), q_\nu)| \leq 1$ ; if  $m < m_0$ , then  $|(P_\nu(x), q_\nu)| \leq m(1 + |x|^2)^{1/2} < m_0(1 + |x|^2)^{1/2}$ . It follows that  $\{|(P_\nu(x), q_\nu)|\}$  is a bounded sequence, and hence  $x \in \text{Conv}(f)$ .

Now suppose that  $x \notin E$ . Then for each positive integer  $m$ , there is a positive integer  $k(m)$  such that  $|(h_{m, k(m)}(x), q_{m, k(m)})| > m/2$ . The sequence  $\{|(h_{m, k(m)}(x), q_{m, k(m)})|\}_{m=1}^\infty$  is unbounded and is a rearranged subsequence of  $\{|(P_\nu(x), q_\nu)|\}$ . It follows that  $\{|(P_\nu(x), q_\nu)|\}$  is an unbounded sequence, and hence  $x \notin \text{Conv}(f)$ . Therefore  $E = \text{Conv}(f)$ , and  $E$  is a convergence set.  $\square$

**Theorem 4.5.** *Every countable union of closed complete pluripolar sets in  $\mathbb{C}^n$  is a convergence set.*

*Proof.* This is a consequence of Theorems 4.4 and 3.9.  $\square$

**Corollary 4.6.** *Every countable union of proper analytic varieties in  $\mathbb{C}^n$  is a convergence set.*

**Corollary 4.7.** *Every countable set in  $\mathbb{C}^n$  is a convergence set.*

**Corollary 4.8.** *A subset of  $\mathbb{C}$  is a convergence set if and only if it is an  $F_\sigma$  polar set.*

*Proof.* This is because each closed polar set in  $\mathbb{C}$  is a complete polar set.  $\square$

## 5. CONVERGENCE SETS IN PROJECTIVE SPACES

For a formal power series  $f(x_1, \dots, x_n) = f(x) \in \mathbb{C}[[x_1, \dots, x_n]]$  and for  $x \in \mathbb{C}^n$ , let  $f_x(t) = f(x_1 t, \dots, x_n t) \in \mathbb{C}[[t]]$ . Since for  $\lambda \in \mathbb{C}$ ,  $\lambda \neq 0$ , the series  $f_x$  and  $f_{\lambda x}$  converge or diverge together, the convergence set of  $f$  (i.e. the set of  $x$  for which  $f_x$  converges) can be identified with a subset of the projective space  $\mathbb{P}^{n-1}$ .

For a non-zero member  $x$  in  $\mathbb{C}^n$ ,  $[x]$  denotes its image in  $\mathbb{P}^{n-1}$ . For a subset  $E$  of  $\mathbb{P}^{n-1}$ , put  $\tilde{E} = \{x \in \mathbb{C}^n : [x] \in E\}$ .

The (projective) convergence set of  $f$  is defined to be

$$\text{Conv}_p(f) = \{[x] \in \mathbb{P}^{n-1} : f_x \text{ converges}\}.$$

**Definition 5.1.** *A subset  $E$  of  $\mathbb{P}^{n-1}$  is said to be a convergence set in  $\mathbb{P}^{n-1}$  if  $E = \text{Conv}_p(f)$  for some divergent series  $f(x_1, \dots, x_n)$ .*

Let  $E$  be a non-empty closed set in  $\mathbb{P}^{n-1}$ . Define  $\Psi_E : \mathbb{P}^{n-1} \rightarrow [0, \infty]$  by  $\Psi_E([x]) = \sup\{|h(x)|^{1/q}/|x| : (h, q) \in \Gamma(\mathbb{C}^n), \|(h, q)\|_{\tilde{E}} \leq 1\}$ . The  $G$ -hull of  $E$  is  $\hat{E}^G = \{u \in \mathbb{P}^n : \Psi_E(u) < \infty\}$ . The  $G$ -hull of the empty set is defined to be the empty set. If  $E$  is non-pluripolar, then  $\hat{E}^G = \mathbb{P}^{n-1}$ . If  $E$  is pluripolar, then  $\hat{E}^G$  is an  $F_\sigma$  pluripolar set.

Recall that there are no non-constant plurisubharmonic functions on  $\mathbb{P}^{n-1}$ .

**Definition 5.2.** *A pluripolar set  $E$  in  $\mathbb{P}^{n-1}$  is said to be complete if there is a function  $h \in H(\mathbb{C}^n)$  such that  $E = \{[x] \in \mathbb{P}^{n-1} : h(x) = 0\}$ . A pluripolar set  $F$  in  $\mathbb{P}^{n-1}$  is said to be  $G$ -complete if  $F = \hat{E}^G$  for some closed pluripolar set  $E$ .*

The proofs of the following two theorems are very similar to those of Theorems 4.2 and 4.3, and hence are omitted.

**Theorem 5.3.** *Let  $E$  be a convergence set in  $\mathbb{P}^{n-1}$ . Then  $E$  is a countable union of  $G$ -complete pluripolar sets. Hence  $E$  is an  $F_\sigma$  pluripolar set.*

**Theorem 5.4.** *Every  $G$ -complete pluripolar set in  $\mathbb{P}^{n-1}$  is a convergence set.*

The set  $\Pi$  of all hyperplanes in  $\mathbb{P}^{n-1}$  is naturally isomorphic to  $\mathbb{P}^{n-1}$ . Each  $\Omega \in \Pi$  is isomorphic to  $\mathbb{P}^{n-2}$ , and its complement in  $\mathbb{P}^{n-1}$  is isomorphic to  $\mathbb{C}^{n-1}$ . For any two hyperplanes in  $\mathbb{P}^{n-1}$ , there is a unitary transformation that maps one to the other.

Fix a positive number  $M$ . Let

$$\begin{aligned} S_1 &= \{[1, 0, \dots, 0]\}, \text{ and for } k = 2, \dots, n, \\ S_k &= \{[x] : |x_1|^2 + \dots + |x_{k-1}|^2 \leq M^2 |x_k|^2, x_{k+1} = \dots = x_n = 0\}. \end{aligned}$$

Put

$$(7) \quad K_M = \cup_{k=1}^n S_k.$$

Then  $\{K_m\}$  is an ascending sequence of closed sets with  $\mathbb{P}^{n-1} = \cup_{m=1}^\infty K_m$ .

**Definition 5.5.** *A subset  $E$  of  $\mathbb{P}^{n-1}$  is said to be non-occupying if there exists  $\Omega \in \Pi$  such that  $E \cap \Omega = \emptyset$ .*

**Lemma 5.6.** *If  $K$  is a closed non-occupying subset of  $\mathbb{P}^{n-1}$  and if  $u \in \mathbb{P}^{n-1}$ , then  $K \cup \{u\}$  is non-occupying.*

*Proof.* Let  $R = \{V \in \Pi : V \cap K = \emptyset\}$  and  $S = \{V \in \Pi : u \in V\}$ . Then  $R$  is a non-empty open set in  $\Pi$  and  $S$  is a hyperplane in  $\Pi$ . Thus  $R \setminus S$  is non-empty.  $\square$

**Lemma 5.7.** *For each  $M > 0$ , the set  $K_M$  is non-occupying.*

*Proof.* Let  $e_1, \dots, e_n$  be the standard basis of  $\mathbb{C}^n$ , and let  $\varepsilon$  be a sufficiently small positive number. Let  $v_j = e_j + \varepsilon e_{j+1}$  for  $j = 1, \dots, n-1$ . Put  $V_j = \text{span}(v_1, \dots, v_j)$  for  $j = 1, \dots, n-1$ , and  $V = V_{n-1}$ . Also, let  $W_j = \text{span}(e_1, \dots, e_j)$ . Note that

$$V \cap W_j \subset V_{j-1}, \text{ for } j \geq 2.$$

Since  $S_j \subset W_j$ , it follows that  $V \cap S_j \subset V_{j-1}$  for  $j \geq 2$ . It is clear that  $V \cap S_1 = \emptyset$ . For  $j \geq 2$  and for sufficiently small  $\varepsilon$ , since  $W_{j-1} \cap S_j = \emptyset$ , and since  $V_{j-1}$  is close to  $W_{j-1}$ , we see that  $V_{j-1} \cap S_j = \emptyset$ . It follows that

$$V \cap K_M = \cup_{j=1}^n (V \cap S_j) \subset \cup_{j=2}^n (V_{j-1} \cap S_j) = \emptyset.$$

Therefore  $K_M$  does not intersect the hyperplane  $V$ .  $\square$

**Definition 5.8.** *A pluripolar set  $E$  in  $\mathbb{P}^{n-1}$  is said to be  $J$ -complete if for each hyperplane  $V$ ,  $E \setminus V$  is  $J$ -complete in  $\mathbb{P}^{n-1} \setminus V$ . The set  $E$  is said to be globally  $J$ -complete if for each  $[x] \in \mathbb{P}^{n-1} \setminus E$ ,  $T_H(x, \tilde{E}) > 0$ .*

It is clear that each  $J$ -complete pluripolar set is closed, and that each globally  $J$ -complete pluripolar set is  $J$ -complete.

The proof of the following proposition is very similar to that of Proposition 3.7, and hence is omitted.

**Proposition 5.9.** *An intersection of (globally)  $J$ -complete pluripolar sets in  $\mathbb{P}^{n-1}$  is (globally)  $J$ -complete. A finite union of (globally)  $J$ -complete pluripolar sets in  $\mathbb{P}^{n-1}$  is (globally)  $J$ -complete.*

**Proposition 5.10.** *Let  $E \subset \mathbb{P}^{n-1}$  be the zero locus of a continuous function  $h \in H(\mathbb{C}^n)$ . Then  $E$  is a globally  $J$ -complete pluripolar set in  $\mathbb{P}^{n-1}$ .*

*Proof.* This is a consequence of Lemma 3.2.  $\square$

**Theorem 5.11.** *Every closed complete pluripolar set in  $\mathbb{P}^{n-1}$  is  $J$ -complete.*

*Proof.* This is a consequence of Theorem 3.9.  $\square$

**Proposition 5.12.** *A proper algebraic variety in  $\mathbb{P}^{n-1}$  is a global  $J$ -complete pluripolar set.*

*Proof.* Let  $E$  be a proper algebraic variety in  $\mathbb{P}^{n-1}$ . Then there are members  $(h_j, q_j)$  of  $\Gamma(\mathbb{C}^n)$ ,  $j = 1, \dots, k$ , such that

$$E = \{[x] \in \mathbb{P}^{n-1} : h_1(x) = \dots = h_k(x) = 0\}.$$

Let  $h = \sum_{j=1}^k |h_j|^{1/q_j}$ . Then  $h \in H(\mathbb{C}^n)$ ,  $h$  is continuous, and  $E = \{h = 0\}$ . By Proposition 5.10,  $E$  is globally  $J$ -complete.  $\square$

For  $[x] \in \mathbb{P}^{n-1}$  and  $S \subset \mathbb{P}^{n-1}$ , we define  $T_H([x], S)$  to be  $T_H(x, \tilde{S})$ . If  $W$  is a hyperplane in  $\mathbb{P}^{n-1}$ , and if  $z$  and  $S$  lie in  $\mathbb{P}^{n-1} \setminus W \cong \mathbb{C}^{n-1}$ , we observe that  $T_H(z, S) = 0$  if and only if  $T_L(z, S) = 0$ .

**Lemma 5.13.** *Let  $E$  be a  $J$ -complete pluripolar set in  $\mathbb{P}^{n-1}$ , let  $K$  be a non-occupying closed set in  $\mathbb{P}^{n-1}$ , let  $[y] \in \mathbb{P}^{n-1} \setminus E$ , and let  $m$  be a real number  $\geq 1$ . Then there exists an  $(h, q) \in \Gamma(\mathbb{C}^n)$  such that*

$$\|(h, q)\| \leq m, \quad \|(h, q)\|_{E \cap K} \leq 1, \quad \|(h(y), q)\| > m/2.$$

*Proof.* By Lemma 5.6,  $K \cup \{[y]\}$  is non-occupying, hence there is a hyperplane  $V$  such that  $\Omega := \mathbb{P}^{n-1} \setminus V \supset (K \cup \{[y]\})$ . Since  $E \cap \Omega$  is a  $J$ -complete pluripolar set in  $\Omega$ , we see that  $T_L([y], E \cap K) > 0$ . It follows that  $T_H([y], E \cap K) > 0$ . Thus there is a positive number  $r < 1$  such that  $\tau_H(x, F, r) = 0$ , i.e.,

$$\inf\{\|(p, v)\|_{E \cap K} : (p, v) \in \Gamma(\mathbb{C}^n), \|(p(y), v)\| \geq r, \|(p, v)\| \leq 1\} = 0.$$

Choose a positive rational number  $\beta = a/b < 1$ , where  $a, b$  are positive integers, such that  $(r/m)^\beta > 1/2$ . There is a  $(p, v) \in \Gamma(\mathbb{C}^n)$  such that

$$\|(p, v)\|_{E \cap K} < m^{-1/\beta}, \quad \|(p(y), v)\| \geq r, \quad \|(p, v)\| \leq 1.$$

Let  $u = \bar{y}/|y|$ . Then  $u$  is a unit vector, and  $\langle y, u \rangle := u_1 y_1 + \cdots + u_n y_n = |y|$ . Put  $h(x) = p(x)^a (m \langle x, u \rangle)^{v(b-a)}$ , and  $q = bv$ . Then  $(h, q) \in \Gamma(\mathbb{C}^n)$ , and  $\|(h(x), q)\| = \|(p(x), v)\|^\beta m^{1-\beta}$ . We have, for all  $x \in \mathbb{C}^n$ ,

$$\|(h(x), q)\| \leq m^{1-\beta} \leq m,$$

$$\|(h, q)\|_{E \cap K} < m^{-1} m^{1-\beta} = m^{-\beta} \leq 1,$$

and

$$\|(h(y), q)\| \geq r^\beta m^{1-\beta} = (r/m)^\beta m > m/2.$$

□

**Theorem 5.14.** *Let  $E$  be a countable union of  $J$ -complete pluripolar sets in  $\mathbb{P}^{n-1}$ . Then  $E$  is a convergence set.*

*Proof.* As in the proof of Theorem 4.4, it is enough to construct a sequence  $\{(P_\nu, q_\nu)\}_{\nu=1}^\infty$  in  $\Gamma(\mathbb{C}^n)$ , with  $\{q_\nu\}$  strictly increasing, such that  $[x] \in E$  if and only if the sequence  $\{\|(P_\nu(x), q_\nu)\|\}$  is bounded, because then  $E$  is the convergence set of  $f(x) = \sum_\nu P_\nu(x)$ .

Since, by Proposition 5.9, the union of a finite number of  $J$ -complete pluripolar sets is  $J$ -complete, we can assume that  $E = \cup E_m$ , where  $\{E_m\}$  is an ascending sequence of  $J$ -complete pluripolar sets in  $\mathbb{P}^{n-1}$ . Recall that  $\mathbb{P}^{n-1} = \cup K_m$ , where  $K_m, m = 1, 2, 3, \dots$ , is the ascending sequence of closed non-occupying sets in  $\mathbb{P}^{n-1}$  defined in (7). For each positive integer  $m$ , we shall construct a sequence  $\{(h_{mk}, q_{mk})\}_{k=1}^\infty$  in  $\Gamma(\mathbb{C}^n)$  such that

- (i)  $\|(h_{mk}, q_{mk})\|_{K_m \cap E_m} \leq 1$ ,
- (ii)  $\|(h_{mk}, q_{mk})\| \leq m$ ,
- (iii)  $\cup_{k=1}^\infty \{[x] \in \mathbb{P}^{n-1} : \|(h_{mk}(x), q_{mk})\| > m/2\} \supset \mathbb{P}^{n-1} \setminus E_m$ .

Fix  $m$  and suppose that  $[y] \in \mathbb{P}^{n-1} \setminus E_m$ . By Lemma 5.13, there exists an  $(h_{[y]}, q_{[y]}) \in \Gamma(\mathbb{C}^n)$  such that

$$\|(h_{[y]}, q_{[y]})\| \leq m, \quad \|(h_{[y]}, q_{[y]})\|_{E_m \cap K_m} \leq 1, \quad \|(h_{[y]}(y), q_{[y]})\| > m/2.$$

Put  $U_{[y]} := \{[x] : \|(h_{[y]}(x), q_{[y]})\| > m/2\}$ . Then  $U_{[y]}$  is an open neighborhood of  $[y]$ . Since the set  $\mathbb{P}^{n-1} \setminus E_m$  is open, the open cover  $\{U_{[y]} : [y] \in \mathbb{P}^{n-1} \setminus E_m\}$  of  $\mathbb{P}^{n-1} \setminus E_m$  contains a countable subcover  $\{U_{[y_k]} : k = 1, 2, \dots\}$ . Put  $(h_{mk}, q_{mk}) = (h_{[y_k]}, q_{[y_k]})$ . Then the sequence  $\{(h_{mk}, q_{mk})\}_{k=1}^\infty$  satisfies (i), (ii) and (iii).

Let  $\{(P_\nu, q_\nu)\}$  be a sequence obtained by arranging  $\{(h_{mk}, q_{mk})\}$  as a single sequence. By raising  $\{(P_\nu, q_\nu)\}$  to suitable powers, we assume



that  $\{q_\nu\}$  is a strictly increasing sequence. Now (i), (ii) and (iii) imply that the sequence  $\{\|(P_\nu(x), q_\nu)\|\}$  is bounded if and only if  $[x] \in E$ .  $\square$

**Theorem 5.15.** *Every countable union of closed complete pluripolar sets in  $\mathbb{P}^{n-1}$  is a convergence set.*

*Proof.* This is a consequence of Theorems 5.14 and 5.11.  $\square$

**Corollary 5.16.** *Every countable union of proper algebraic varieties in  $\mathbb{P}^{n-1}$  is a convergence set.*

**Corollary 5.17.** *Every countable set in  $\mathbb{P}^{n-1}$  is a convergence set.*

**Corollary 5.18.** *A subset of  $\mathbb{P}^1$  is a convergence set if and only if it is an  $F_\sigma$  polar set.*

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